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THE SOLUTION OF INVERSE PROBLEMS IN UNDERWATER GEOACOUSTICS BY OPTIMAL CONTROL METHODS

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Abstract - The propagation of acoustic signals in a waveguide can be efficiently modelled by a parabolic approximation of the wave equation. Small and large angle parabolic approximation based on rational Padé approximations are used together with a local and a non local (physically more realistic) impedance boundary condition on the sea floor. The inverse problem consists of computing the coefficients of the impedance condition (in order to reconstruct the acoustic field in the waveguide) from measurements taken along a vertical hydrophone array placed at a given range. The source signal (at range zero) is known, and a zero pressure boundary condition is applied on the sea surface. The optimal control method for this problem is set up as follows. First, define a least-squares cost function (usually the mismatch between the known measurements and the simulated field), constrained by the parabolic wave equation. Take the variation of the cost function with respect to the control function (that appears in the impedance boundary condition). Obtain an explicit representation of the gradient of the cost function by introducing an adjoint boundary value problem. Finally, use this exact gradient in a gradient-based optimization loop. The optimal control, thus computed, solves the inverse problem with high accuracy and is computationally inexpensive when compared to the classical signal theory approaches currently used in acoustics. We present the physical problem, the steps of the mathematical modelling and the optimal control formulation. Then we discuss the numerical approximation using implicit finite differences and a conjugate gradient optimization algorithm with linesearch. Finally we show results of numerical simulations, ranging from synthetic examples and benchmarks to comparisons with environments based on real measurements obtained from sea trials.

1. INTRODUCTION

The acoustic field in a waveguide can be modelled by a paraxial approximation of the wave equation. If we decompose the acoustic pressure $p(r,z) = u(r,z)H_0^{(1)}(k_0 r)$, where $H_0^{(1)}$ is the Hankel function, then the field satisfies (see [1])

$$2ik_0\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2-1)u = 0 \tag{1}$$

or

$$2ik_0\left[1+\frac{1}{4}(n^2-1)\right]\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2-1)\psi + \frac{i}{2k_0}\frac{\partial^3 \psi}{\partial r \partial z^2} = 0 \quad , 0<r<R,\, 0<Z<H. \tag{2}$$
$$\psi(0,z) = S(z),\, \psi(0,r) = 0,\, G_2(\psi(r,H),\gamma) = 0.$$

where $k_0 = \omega/c_0$, $n = c_0/c$, ω is a fixed frequency, $c = c(z)$ is the vertical sound speed profile and c_0 is a reference sound speed profile. The first equation is known as the small-angle parabolic equation (SPE), the second as the wide-angle parabolic equation (WAPE). We suppose that the boundary condition on the seafloor $z = H$ is given by

$$\frac{\partial u}{\partial z} + i\gamma u = 0 \tag{3}$$

where γ is a complex-valued function depending on r and represents an impedance, or by (see [10])

$$\left\{\frac{\partial}{\partial z} - i\mathcal{B}\right\}\psi\left[(J+1)\Delta r, z_b\right] = i\mathcal{B}\sum_{j=1}^{J+1} g_{0,j}\,\psi\left[(J+1-j)\Delta r, z_b\right] \tag{4}$$

where $\mathcal{B} = \frac{\rho_b}{\rho_w} k_0 \sqrt{N_b^2 - 1 + \nu^2}$, $k_0 = \frac{\omega}{c_0}$, $N_b = n_b[1 + i\alpha]$, $n_b = \frac{c_0}{c_b}$, $\nu^2 = \frac{4i}{k_0 \Delta r}$, α is an attenuation, ρ is density, c is sound speed, c_0 is a reference sound speed, subscript b denotes the sea bottom and subscript w denotes the water layer. This boundary condition provides an excellent model of the sea bottom, and coupled with the WAPE gives a highly acceptable physical model.

The boundary value problem is completed by taking a zero pressure condition on the sea's surface and a known initial condition (signal source) at $r = 0$. The problem is then solved for $0 \leq r \leq R$ a given range, and $0 \leq z \leq H$ a given depth.

Let us suppose now that we measure $u_d(R, z)$ on a line of hydrophones situated at the range $r = R$. From these measurements we would like to recover the boundary coefficient $\gamma(r)$ in (3) or $\gamma \doteq (\rho, c, \alpha)$ in (4) that play the role of the *control functions*. Let the cost function be (we must add regularization terms – see [2, 4] – on γ in order to make the optimization problem less ill-posed by restoring the continuous dependence)

$$J(\gamma) = \frac{1}{2} \int_{z=0}^{z=H} |u(\gamma; R, z) - u_d(R, z)|^2 + \text{ regularization terms}, \tag{5}$$

where $u(\gamma; r, z)$ is obtained from solving the paraxial equation (1) or (2) with the boundary condition (3) or (4) and $u_d(R, z)$ is the desired or measured solution. The inverse problem can then be stated as follows: for given measurements $u_d(R, z)$ of an acoustic field, find the complex coefficient $\gamma(r)$ of the impedance boundary condition that minimizes the cost function $J(\gamma)$. In other words, find $\gamma* \in \mathcal{G}$, the optimal control, such that

$$J(\gamma*) = \inf_{\gamma \in \mathcal{G},} J(\gamma),$$

where $\mathcal{G}$ is the set of admissible controls.

2. THE ADJOINT STATE METHOD

The adjoint method is a technique used for calculating an *exact* expression of the gradient of a cost function that depends *implicitly* on the control variables with respect to which we must differentiate. The method avoids the costly calculation of the derivative of the implicit function and is thus extremely well suited to *optimal control* problems. The gradient can be calculated by means of a *Lagrange multiplier* or an *adjoint state*.

Before discussing the method itself, we introduce some general definitions and notation for *inverse problems*. A more detailed review of the approach can be found in [2] where the important issue of regularization is also discussed (see also [4] for an application to the geoacousic inversion problem). In [8] one can find a discussion of the Fréchet derivatives that we use to derive our functional gradient.

Inverse problems

In order to give an abstract formulation of the class of inverse problems, we introduce three Hilbert spaces as well as mappings between these spaces. The spaces are:

- the *model* (control or parameter) space M ;
- the *state* space U ;
- the *data* (or observation) space D ;

The state space U enables us to describe explicitly the dependence between the control (parameters) and the data. Nonetheless, the existence of the state does not dispense us of introducing the observation, since the state is in general not measurable. Two mappings (equations) give the relationships between these three spaces.

The state equation links implicitly the control and the state. We write

$$F(a, u) = 0, \quad a \in M,\ u \in U,\ F(a, u) : M \times U \to Z \tag{6}$$

where Z is another Hilbert space. We suppose that there exists a subspace $M_{\text{ad}} \subset M$ (the space of "admissible" controls) such that for all $a \in M_{\text{ad}}$, F locally defines a unique state $u = u_a$.

It will be useful to denote the solution of the state equation (6) as

$$u = S(a) = u_a. \tag{7}$$

The observation equation extracts from the state, the part that corresponds to the measurements. This will often be an injection due to the inherent difficulty of measuring the state at a large number of points. It is written as

$$d = Hu, \quad u \in U. \tag{8}$$

We have made the simplifying assumption that the observation is a linear operator, independent of the control. The extension to a more general situation is not difficult.

If we substitute the solution of (6) into (8), we obtain the mapping that relates the control (or the parameter) to the observation. We write

$$d = \Phi(a) = H\big(S(a)\big) = H(u_a) \tag{9}$$

The inverse problem is then: given an observation d_{obs}, solve the equation

$$\Phi(a) = d_{\text{obs}} \tag{10}$$

for the control (or parameter) a.

Least-Squares Formulation

In most cases the mapping Φ is defined implicitly. It is also nonlinear, even if the state and observation equations are linear. This clearly makes it difficult to solve the inverse problem.

The problems (9) or (10) may not have a solution, and even if they have one, the inverse mapping is not necessarily continuous. We are thus led to introduce a weaker formulation that has turned out to be very effective. We *replace* equation (9) or (10) by the following *minimization* problem:

$$\text{minimize } J(a) = \frac{1}{2}\,\|\Phi(a) - d_{\text{obs}}\|_D^2 \text{ for } a \in M_{\text{ad}}. \tag{11}$$

This formulation is called the output *least-squares method* for the *cost function*, or functional, J. It is very important to understand how this functional "functions". The observation is given once and for all, then to evaluate the functional J at a parameter a, we start by solving the state equation (6), followed by the observation equation (8), and finally we compare the *simulated* observation to the measured one.

This reformulation will not magically transform an ill-posed problem into a well-posed one. However, it re-establishes the existence. Indeed, even if no solution exists for the equation (9), the minimization problem will necessarily have a solution since the cost function J is positive. On the other hand, nothing guarantees that the minimum will be attained at a point $a \in M_{\text{ad}}$. Another important question is that of uniqueness which is related to the convexity of J and once more, there are no guarantees. The formulation (11) does of course have numerous qualities:

- It gives a systematic method for the formulation of inverse and optimal control problems.
- In certain cases we can prove the necessary properties of J for the existence of a minimum.
- It enables us to regularize the problem by means of a family of well-posed problems whose limit solution converges to the solution of the original problem.
- There are robust, well studied numerical methods for solving optimization problems.
- Under reasonable hypotheses on the data, the functional J is differentiable and can thus be minimized by local gradient-based optimization methods.

The Difficulties of Inverse Problems

The difficulties arise from a combination of factors.

- The cost function is in general *non-convex.* This leads to the existence of local minima and the optimization algorithms will experience difficulties to converge to a "good" global minimum. We can (and often do) obtain *non-physical* solutions.
- The inverse problem can be *under-determined* due to a lack of data. Here, once again, we can have several solutions producing the same observations.
- The lack of continuity produces a *discontinuity.* Noise in the data can prevent us from being able to solve the problem. This can be (partially) dealt with by penalization techniques.

Methods for computing the gradient

In this section we consider a method for computing the gradient of the *least-squares* type cost function

$$J(a) = \frac{1}{2} \|\Phi(a) - d_{\text{obs}}\|_D^2 \tag{12}$$

where the nonlinear functional Φ is defined by the solution of the *state equation*

$$F(a, u) = 0,$$

whose solution is u_a, then by extracting from u_a the *observation* equation

$$\Phi(a) = H u_a. \tag{13}$$

The difficulty clearly lies in the calculation of the derivative of the (implicit) function $a \to u_a$. We detail the *adjoint state method* that permits this calculation at a cost that is independent of the number of control parameters.

The sensitivity function

This is the most natural method for calculating the gradient of J. We differentiate the state equation explicitly with respect to the parameter a, then we use the rules for the differentiation of a composite function. We emphasize that this method gives an *exact* result.

Recall that

$$\nabla J(a) = \Phi'(a)^*(\Phi(a) - d_{\text{obs}}). \tag{14}$$

First we calculate the Jacobian of Φ. Since Φ is defined implicitly by the solution of the state equation (6) and the observation equation (13), we must use the Implicit Function Theorem (or more precisely, its corollary that permits one to calculate the differential of the implicit mapping once we know that it is differentiable). This result tells us that we can obtain the Jacobian of Φ by differentiating the state equation

$$\partial_u F(a, u)\delta u + \partial_a F(a, u)\delta a = 0, \tag{15}$$

solving the (linear) equation obtained, and composing with the (derivative of) the observation (that we have supposed linear). Thus

$$\delta u/\delta a = \Phi'(a) = -H \left(\partial_u F(a, u)\right)^{-1} \partial_a F(a, u) \tag{16}$$

Regrouping the equations (14) and (16), we finally obtain the gradient of J,

$$\nabla J(a) = \left(\Phi'(a)\right)^* \left(Hu(a) - d_{\text{obs}}\right). \tag{17}$$

The main disadvantage of this method lies in the fact that the computation of δu requires the solution of a (linearized) state equation for each value of δa. After passing to finite dimension, this means that the calculation of each partial derivative $\partial J/\partial a_j$ requires the solution of an equation like (15). The cost of the gradient computation is thus *proportional* to the number of parameters. This number can often be very large: a few hundred, or even thousands. We will see below that the main advantage of the adjoint state method is the possibility to achieve this computation at a cost proportional to that of a *single* linearized equation solution, and in particular, *independently* of the number of parameters.

On the other hand, the sensitivity function method provides more than just the gradient, but also gives the Jacobian of Φ. The Gauss-Newton method needs this Jacobian. If the number of parameters is not too large, the Gauss-Newton method, by calculating gradients according to (17) can be more economical than a quasi-Newton method with computation of the gradient by the adjoint state.

The adjoint state method

We have already seen that the sensitivity function method provides more than just the gradient of J. If we need only the gradient, we can rearrange the calculation leading to (17) in order to avoid calculating the full Jacobian. By substituting (16) in (17), and transposing the product, we obtain

$$\nabla J(a) = -\left[\partial_a F(a, u)^* (\partial_u F(a, u)^*)^{-1} H^*\right] \left[Hu(a) - d_{\text{obs}}\right]. \tag{18}$$

The apparently trivial remark that will enable us to simplify the calculation, is that it is possible to parenthesize this expression differently,

$$\nabla J(a) = -\left[\partial_a F(a, u)\right]^* \left[(\partial_u F(a, u)^*)^{-1} \left(H^* \left(Hu(a) - d_{\text{obs}}\right)\right)\right]. \tag{19}$$

It is convenient to give a name to the quantity inside the second parenthesis, and to introduce the vector p, solution of

$$\partial_u F(a,u)^* p = -H^* \left(Hu(a) - d_{\text{obs}} \right). \tag{20}$$

We call this equation the *adjoint equation*, and p is the *adjoint state.* Once we have solved this equation, the gradient is directly calculated from

$$\nabla J(a) = \partial_a F(a,u)^* p. \tag{21}$$

We summarize all of this in a theorem. See [2] for the proof in a general setting.

Theorem 1 *If p is the solution of the adjoint equation (20), then the gradient of J at the point a is given by (21), where $u = u(a)$ is the solution of the state equation (6) corresponding to a.*

Theorem 1 is of great importance. It provides the general framework on which the adjoint state method is built. Since it can sometimes be difficult to apply directly in practice, we will propose a more simple method for achieving the same result. Indeed, it can be difficult in a real case to identify the different adjoints concerned, and even the operator F itself. This will be particularly true for evolution problems.

Calculation of the adjoint state by the lagrangian

The previous paragraph showed how to calculate the gradient of a cost function by solving only two equations: the state equation, followed by the adjoint equation. The operations of the formula (21) are typically very simple, but the implementation is not as easy. We now present a technique that leads to the same result, but turns out to be more flexible.

The method is based on what can be considered as a computational trick. We start by claiming that the variables a and u vary independently, and we consider the state equation as a constraint. Under these conditions, as we saw before, it is natural to introduce a Lagrangian. In our case, the Lagrangian is written as (at least formally, since we do not have a finite number of constraints)

$$\mathcal{L}(a,\, u,\, p) = \frac{1}{2} \left\| Hu - d_{\text{obs}} \right\| + \left\langle p,\, F(a,\, u) \right\rangle. \tag{22}$$

The fundamental remark (once again apparently trivial) is that, if u satisfies the state equation corresponding to the parameter a, then we have the identity

$$\mathcal{L}(a,\, u(a),\, p) = J(a),\ \forall p \in Z$$

since $F(a,\, u) = 0$. Differentiating this relation, we obtain

$$J'(a)\delta a = \partial_a \mathcal{L}(a,\, u)\, \delta a + \partial_u \mathcal{L}(a,u)\, \partial_a u(a) \delta a. \tag{23}$$

The difficult part is then to calculate $\partial_a u(a)$. If we can *choose* $p \in Z$ such that this last term disappears, we will have a simple expression for the derivative of J. To this end, we will suppose that $\delta u = \partial_a u(a)\, \delta a$ is an independent quantity, and we require that the operator $\delta u \to \partial_u \mathcal{L}(a,u)$ disappears. We thus define the abstract *adjoint equation* by

$$\partial_u \mathcal{L}(a,\, u(a))\, \delta u = 0, \quad \forall \delta u \in U, \tag{24}$$

and by expanding the expression for $\partial_u \mathcal{L}$

$$\langle H\delta u, Hu(a) - d_{\text{obs}} \rangle + \langle p, \partial_u F(a, u(a)) \rangle\ \delta u = 0, \quad \forall \delta u \in U. \tag{25}$$

We have again found precisely the adjoint equation (20). Then the differential of J is calculated by the formula

$$J'(a)\delta a = \langle p, \partial_a F(a, u(a)) \rangle\, \delta a \tag{26}$$

and we observe that in fact we obtain the gradient of J,

$$\nabla J(a) = (\partial_a F(a, u(a)))^* p, \tag{27}$$

that is identical to (21).

Remark 2 *In practice, the most useful form of the adjoint equation is the "variational" formulation (25). In fact, as we have already pointed out, it is not always convenient to calculate the adjoint operators. By contrast, it is always simple (we will see below) to start out from the form (25), and to manipulate it (by integration or summation by parts), in order to reach an explicit adjoint equation. In the same way, it is often more convenient to start out from equation (26) and to manipulate it in order to identify the gradient than to use formula (27) as it is.*

We resume the main steps in a theorem.

Theorem 3 *The computation of the gradient of the functional (11) is achieved by the following steps:*

1. *Define the Lagrangian by (22).*
2. *Solve the (variational) adjoint equation (25) that determines the adjoint state p.*
3. *The differential of J is given by (26) and enables the identification of the gradient of J.*

3. APPLICATION OF THE ADJOINT METHOD TO THE INVERSE PROBLEM OF UNDERWATER GEOACOUSTICS

Computation of the gradient of the cost function

For simplicity of the presentation we consider the small angle parabolic equation (1) with the local boundary condition (3). The extension to the physically interesting case of the wide angle equation and the nonlocal boundary condition can be found in [3, 6]. We start by taking the variation of J in (5)

$$\begin{aligned}\lim_{t\to 0}\frac{J(\gamma+t\varphi)-J(\gamma)}{t} &= \frac{1}{2}\lim_{t\to 0}\int_{z=0}^{z=H}\frac{|u(\gamma+t\varphi;R,z)-u_d(R,z)|^2-|u(\gamma;R,z)-u_d(R,z)|^2}{t}\\ &= J'(\gamma,\varphi)=g'(0),\end{aligned}$$

where g is the function defined by

$$g(t)=J(\gamma+t\varphi)$$

and ϕ is a smooth perturbation with compact support. We introduce the real-valued scalar product

$$\langle u,v\rangle = Re(u\overline{v}).$$

This scalar product satisfies

$$\langle u,u\rangle = |u|^2\,,\quad \langle iu,v\rangle = -\langle u,iv\rangle\,,$$
$$\langle u,v\rangle = \langle v,u\rangle\,,\quad \langle \gamma u,v\rangle = \langle u,\overline{\gamma}v\rangle$$

and the derivative formula

$$d|u|^2 = 2\langle u,du\rangle\,.$$

We then have

$$g'(t)=\int_{z,r=R} < u(\gamma+t\varphi)-u_d, \frac{du}{dt}(\gamma+t\varphi) > dz,$$

where $w=\frac{du}{dt}$ is the solution of the *linear tangent model* (we suppose that $n^2=1$ for simplicity)

$$\begin{aligned}2ik_0w_r+w_{zz} &= 0,\\ w(0,z) &= 0,\\ w(r,0) &= 0,\\ w_z(r,H)+i\gamma w(r,H) &= -i\varphi(r)u(\gamma+t\varphi;r,H).\end{aligned}$$

The functional derivative of J is then, with $t=0$,

$$J'(\gamma,\varphi)=\int_{z,r=R} < u-u_d, w > dz. \tag{28}$$

The adjoint state p is written here as

$$\begin{aligned} 2ik_0 p_r + p_{zz} &= 0, \\ p(R,z) &= u - u_d, \\ p(r,0) &= 0, \\ p_z(r,H) - i\overline{\gamma} p(r,H) &= 0. \end{aligned} \tag{29}$$

This equation for p is backward in r, since it is driven by a "terminal" condition at $r = R$, and the boundary condition is still dissipative in spite of the change of sign of γ! Integrating by parts

$$\int_z \int_r < w_r - \frac{i}{2k_0} w_{zz}, p >= 0$$

and using (29) and (28), we obtain

$$\int_{z,r=R} < u - u_d, w >= \frac{1}{2k_0} \int_{r,z=H} < u\varphi, p >,$$

whence the result

$$J'(\gamma, \varphi) = \frac{1}{2k_0} \int_{r,z=H} < \overline{u}p, \varphi >,$$

which can be rewritten as

$$\nabla J = \frac{\overline{u}p}{2k_0}.$$

The gradient method
Once the gradient of the cost function ∇J is known, we can seek a local minimum of $J(\gamma)$. The simplest method for doing this is by steepest descent which uses the update

$$\gamma^{(n+1)} = \gamma^{(n)} - \alpha \nabla J(\gamma^{(n)}),\ \alpha > 0\,,$$

for $n = 0, 1, ...$ until convergence. In order to accelerate the convergence we will use a conjugate gradient method of Fletcher-Reeves or Polak-Ribière type. Here the update is given by

$$\gamma^{(n+1)} = \gamma^{(n)} + \alpha_n d_n\,,$$

where α_n is the step-length that minimises J in the direction d_n, and this direction is computed in two steps:

$$\beta_n = \frac{\nabla J_{n+1}^T \nabla J_{n+1}}{\nabla J_n^T \nabla J_n}$$

$$d_{n+1} = -\nabla J_{n+1} + \beta_{n+1} d_n\,.$$

Finite difference approximation
The direct and adjoint boundary value problems are approximated by an implicit, second-order accurate, Crank-Nicolson finite difference scheme.

4. NUMERICAL RESULTS

Due to lack of space, we only show results for the case of the small angle equation with local boundary condition (1-3). The case of (2-4) is reported elsewhere - see [6]. As are numerical implementations of regularization - see [4].

In Figures 1 and 2 we show the initial acoustic field, the true field and the field obtained after inversion. The inversion succeeds admirably in both the reconstruction of the initial condition at $r = 0$ as well as the principal features of the field. The relative errors in a 2-norm are only a few percent. Note the manifestation of the non-convexity of the cost function for the inverse problem in Figure 2 part (b), where the inverted γ is quite different from its true value and exhibits oscillations. In order to remedy this phenomenon (it being inacceptable to the sonar operators) we perform some regularization. Details can be found in [4]. In addition, as has been shown in [7], the use of multiple frequencies improves the observability thus reducing the ill-posedness/non-convexity and effectively damping out the oscillations.

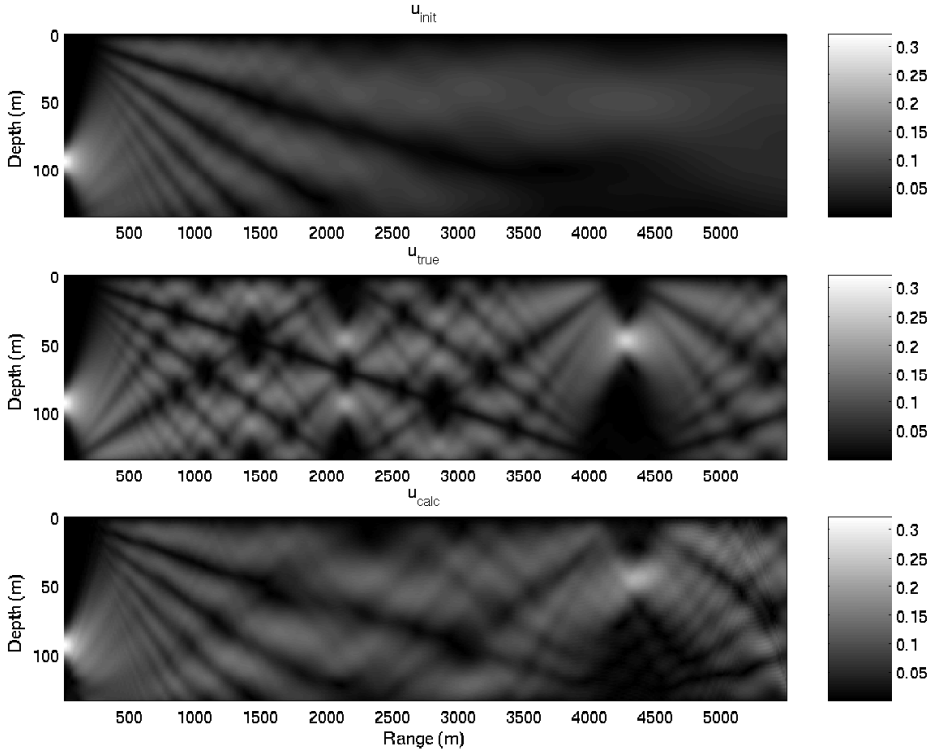

Figure 1: Initial acoustic field (top), true acoustic field (middle) and inverted acoustic field (bottom).

5. CONCLUSIONS

We have formulated an adjoint state method for the solution of inverse problems in underwater geoacoustics. The numerical implementation of this method accurately reconstructs the acoustic field (on condition that we have good initial guesses) and converges in a small number of iterations (5 to 10). Problems of spatial oscillations arise and can be dealt with by suitable regularization techniques and by the use of a multiple frequency cost function. The method is now being applied to realistic, double-layer models of the ocean floor and compares favorably with environments based on real measurement campaigns, such as those found in [9].

REFERENCES

1. F.B. Jensen, W.A. Kuperman, M.B. Porter, and H. Schmidt. *Computational Ocean Acoustics.* American Institute of Physics Press, New York, 1994.

2. C. Kravaris and J. H. Seinfeld. Identification of parameters in distributed parameter systems by regularization. *SIAM J. Control and Optimization* (1985) **23**, pp. 217-241.

3. M. Meyer, J.-P. Hermand and M. Asch, Derivation of the adjoint of a wide angle parabolic equation for acoustic inversion, *Proceedings of the Seventh European Conference on Underwater Acoustics*, ECUA 2004, Delft, The Netherlands, 5-8 July, 2004.

4. M. Meyer, J.-P. Hermand, J.-C. Le Gac, Penalization method for PE adjoint-based inversion of an acoustic field, *Proceedings of the Seventh European Conference on Underwater Acoustics*, ECUA 2004, Delft, The Netherlands, 5-8 July, 2004.

5. M. Meyer, J.-P. Hermand, Reciprocity in acoustic inversion: Time reversal, back wave propagation and adjoint modelling, *2nd workshop on Acoustic inversion methods and experiments for assessment of the shallow water environment,* Ischia, Kluwer Academic Press, 2004.

6. M. Meyer, J.-P. Hermand. Optimal nonlocal boundary control of the wide-angle parabolic equation for inversion of a waveguide acoustic field. *J. Acoust. Soc. Amer.*, submitted (2005).

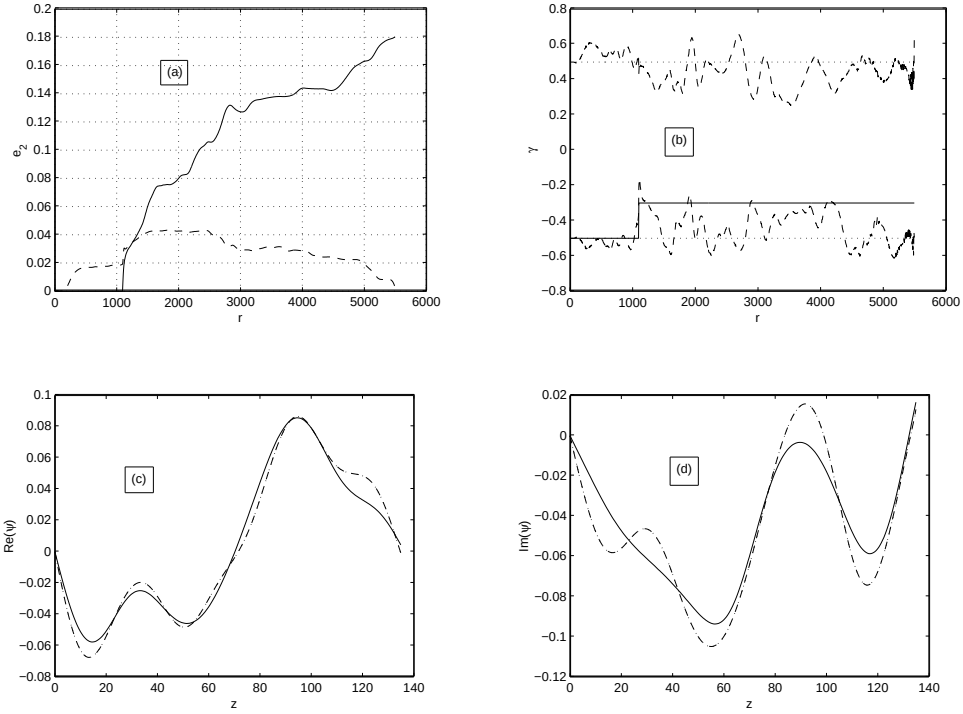

Figure 2: Convergence results for inversion: (a) Relative l_2-error of inital field (solid line, varying from 0 to 0.18 at $r = R$) and final field (dashed line, less than 0.05 for all r) as function of r; (b) initial (solid), true (dotted) and inverted (dashed) values for $\Re(\gamma)$ and $\Im(\gamma)$; (c) and (d) initial (solid), true (dotted) and inverted (dashed) values of the acoustic field at $r = R$. Note that the true and inverted curves in (d) are perfectly superposed and thus indistinguishable.

7. M. Meyer, J.-C. Le Gac, J.-P. Hermand, M. Asch. An iterative multiple frequency adjoint-based inversion algorithm for parabolic-type approximations in ocean acoustics. *Inverse Problems,* submitted (2005).

8. S. J. Norton. Iterative inverse scattering algorithms: Methods of computing Fréchet derivatives. *J. Acoust. Soc. Amer.* (1999) **106**, pp. 2653-2660.

9. M. Siderius and J.-P. Hermand, Yellow Shark Spring 1995: Inversion results from sparse broadband acoustic measurements over a highly range dependent soft clay layer, *J. Acoust. Soc. Amer.* (1999) **106**, pp. 637-651.

10. Yevick D., Thomson D.J., Nonlocal boundary conditions for finite difference parabolic equation solvers, *J. Acoust. Soc. Amer.* (1999) **106**, pp.143-150.